New Solvable Shape-Invariant Potentials for Position-Dependent Effective Mass

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Abstract

Four new exactly solvable, real and shape-invariant potentials associated with a position-dependent effective mass are generated within the concept of shape-invariant potentials using a specific ansatz for superpotential. The accompanying energy spectra of the bound-state and the ground-state wavefunction are obtained algebraically as a function of free parameters and the results are compared with those of others works in the litterature.

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1 Introduction

Physical systems with position-dependent effective mass have received, in recent years, a significant attention due to their relevance in describing the physical properties of various microstructures such as compositionally graded crystals [1], quantum dots [2], semi conductor heterostructures [3], quantum liquids [4] and $^3H-$clusters [5], etc. Recently, a wide number of exact solutions on these topics has increased [6-11]. In theoretical physics,

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various methods and approaches are used including Factorization and Operator methods [12,13], Point canonical transformation methods [14], Supersymmetric quantum mechanical approach [15], Group theory approach [16] and Path integral formalism [17].

The effective mass Schrödinger equation is studied by many authors and its exact solutions (eigenvalues and eigenfunctions) are obtained. Quesne and Tkachuk have established a certain connection between the Schrödinger equation with position-dependent effective mass and the deforming function appearing in the generalized canonical commutation relations. As a consequence, the potential in the deformed Schrödinger equation can be considered as the effective potentials in position-dependent effective mass [18,19].

The main purpose of the present paper is to obtain the energy spectra of the bound-states and the ground-state wavefunctions for four "new" solvable and real potentials (Three-dimensional Harmonic Oscillator, Morse, Pöschl-Teller I and II, and three-dimensional Coulomb) deduced by applying the procedure of Ref. [18] in order to solve a position-dependent effective mass as a deformed shape-invariance condition introduced by Gendenshtein [20] and inspired by the supersymmetric quantum mechanical technique [21]. The one-dimensional potentials thus obtained are Shape-Invariant under parameter translations $\boldsymbol{\lambda}_{i+1} = \boldsymbol{\lambda}_i \pm \boldsymbol{a}$ where $\boldsymbol{\lambda}_i$ and $\boldsymbol{a}$ denote two sets of three parameters, i.e. $\boldsymbol{\lambda}_i = (\lambda_i, \sigma_i, \rho_i)$ and $\boldsymbol{a} = (a, b, c)$ with $i = 0, 1, 2, \ldots$ The new shape-invariance yields a new energy spectra of the bound-states with a non-equidistant spectrum which present some features with those obtained in Ref. [25].

The plan of the present paper is as follow. Factorization, effective potential and Shape-Invariant Potentials are briefly reviewed in section 2. Section 3 will deal with an assumption carried with the superpotential in order to discuss the shape invariance condition leading to the class of potentials and its corresponding "new" energy eigenvalues as well as their ground-state wavefunction. Finally, the last section contain the conclusion.

2 Factorization, Effective potential and Shape Invariant Potentials.

There are several ways to define the kinetic energy operator when the mass is a function of position. Since the mass and momentum operators are

no longer commute, the generalization of the standard Hamiltonian is not trivial. Therefore, the choice of the correct ordering of the operators of the kinetic energy to be Hermitian is indispensable. Defining a general Hermitian effective mass Hamiltonian proposed by von Roos [3]

$$\mathcal{H}_{VR} = \frac{1}{4}\left(m^{\alpha}(x)\,pm^{\beta}(x)\,pm^{\gamma}(x) + m^{\gamma}(x)\,pm^{\beta}(x)\,pm^{\alpha}(x)\right) + V(x)\,, \tag{2.1}$$

where $\hbar = 1,\ \alpha+\beta+\gamma = -1$, and $p = -i\dfrac{d}{dx}$. The limits of the choice of the parameters α, β and γ depend on the physical system. Using the restricted Hamiltonian from the $\alpha = \gamma = 0$ constraint used by BenDaniel and Duke [22], we can write (2.1) as

$$\mathcal{H}_{VR} = -\partial_x U^2(x)\,\partial_x + V(x)\,, \tag{2.2}$$

with $U^2(x) = \dfrac{1}{2m(x)}$. Here we have used abbreviation $\partial_x = \dfrac{d}{dx}$.

The identity found upon the commutation relation

$$\begin{aligned}\left[\partial_x, \sqrt{U(x)}\right] &= \partial_x\sqrt{U(x)} - \sqrt{U(x)}\partial_x \\ &= \frac{1}{2}\frac{\partial_x U(x)}{\sqrt{U(x)}},\end{aligned} \tag{2.3}$$

brings the kinetic term in (2.2) to

$$\begin{aligned}\partial_x U^2(x)\,\partial_x &= \partial_x\sqrt{U(x)}U(x)\,\sqrt{U(x)}\partial_x \\ &= \left[\sqrt{U(x)}\partial_x + \frac{1}{2}\frac{\partial_x U(x)}{\sqrt{U(x)}}\right]U(x)\left[\partial_x\sqrt{U(x)} - \frac{1}{2}\frac{\partial_x U(x)}{\sqrt{U(x)}}\right] \\ &= \left(\sqrt{U(x)}\partial_x\sqrt{U(x)}\right)^2 - \frac{U''(x)\,U(x)}{2} - \frac{U'^2(x)}{4}.\end{aligned} \tag{2.4}$$

where the prime " $'$ " refers to the derivetive of $U(x)$ with respect to x.

The Hamiltonian (2.1) becomes [18,19]

$$\mathcal{H}_{VR} = -\left(\sqrt{U(x)}\partial_x\sqrt{U(x)}\right)^2 + V_{eff}(x)\,, \tag{2.5}$$

where the effective potential $V_{eff}(x)$ is defined following (2.1) and (2.4) as

$$V_{eff}(x) = V(x) + \mathcal{V}_U(x)\,, \tag{2.6}$$

with

$$\mathcal{V}_U(x) = \frac{U''(x)\,U(x)}{2} + \frac{U'^2(x)}{4}. \tag{2.7}$$

In the formalism of supersymmetric quantum mechanics, there are two operators Q and $Q^\dagger$, called supercharges, that satisfy the anti commutation relation $\{Q, Q^\dagger\} = \mathcal{H}_{SS}$, where $\mathcal{H}_{SS}$ is the supersymmetric Hamiltonian [21]. The standard realization of the operators Q and $Q^\dagger$ is $Q = A\sigma_-$ and $Q^\dagger = A^\dagger\sigma_+$ where $A\left(A^\dagger\right)$ and $\sigma_-\left(\sigma_+\right)$ are the bosonic operators and Pauli matrices, respectively. As a consequence of this, the appropriate operators to study the Hamiltonian (2.5) are

$$\begin{aligned} A_{eff} &= \sqrt{U(x)}\partial_x\sqrt{U(x)} + W_{eff}(x), & \text{(2.8.a)} \\ A^\dagger_{eff} &= -\sqrt{U(x)}\partial_x\sqrt{U(x)} + W_{eff}(x). & \text{(2.8.b)} \end{aligned}$$

where $W_{eff}(x)$ is called the effective superpotential.

With this realization, the supersymmetric Hamiltonian (2.5) of the quantum system with position-dependent effective mass takes the form

$$\begin{aligned} \mathcal{H}_{1,eff} &\equiv A^\dagger_{eff}A_{eff} \\ &= -\left(\sqrt{U(x)}\partial_x\sqrt{U(x)}\right)^2 + V_{1,eff}(x), & \text{(2.9.a)} \end{aligned}$$

where

$$V_{1,eff}(x) = W^2_{eff}(x) - U(x)\,W'_{eff}(x), \tag{2.9.b}$$

and

$$\begin{aligned} \mathcal{H}_{2,eff} &\equiv A_{eff}A^\dagger_{eff} \\ &= -\left(\sqrt{U(x)}\partial_x\sqrt{U(x)}\right)^2 + V_{2,eff}(x), & \text{(2.10.a)} \end{aligned}$$

with

$$V_{2,eff}(x) = W^2_{eff}(x) + U(x)\,W'_{eff}(x). \tag{2.10.b}$$

The Hamiltonian $\mathcal{H}_{2,eff}$ is called the supersymmetric partner of $\mathcal{H}_{1,eff}$. It can be easily shown that both $\mathcal{H}_{1,eff}$ and $\mathcal{H}_{2,eff}$ has the same spectrum except for the ground-state, which belongs to $\mathcal{H}_{1,eff}$ [21]. It is obvious that (2.9.b) and (2.10.b) are related by

$$V_{2,eff}(x) = V_{1,eff}(x) + 2U(x)\,W'_{eff}(x). \tag{2.11}$$

Substituting $\mathcal{V}_U(x)$ given by (2.7) as defined in (2.6) into $V_{i,eff}(x)$ with $i = 1, 2$ given by (2.11), we end up with

$$V_2(x) = V_1(x) + 2U(x)\,W'_{eff}(x)\,. \tag{2.12}$$

However, following the paper of Samani and Loran [23], the potential $V_2(x)$ reads as

$$V_2(x) = V_1(x) + 2U(x)\,W'(x) - U(x)\,U''(x)\,, \tag{2.13}$$

and comparing (2.12) to (2.13), we obtain after integration the relationship connecting both the superpotential and the effective superpotential

$$W(x) = W_{eff}(x) + \frac{U'(x)}{2}. \tag{2.14}$$

Inserting (2.14) into (2.9.b) and (2.10.b), we get [23]

$$\begin{aligned}
V_1(x) &= W^2(x) - [U(x)\,W(x)]'\,, && (2.15.a)\\
V_2(x) &= W^2(x) - [U(x)\,W(x)]' + 2U(x)\,W'(x) \\
&\quad -U(x)\,U''(x)\,. && (2.15.b)
\end{aligned}$$

Despite their similar bound-state energy spectra, supersymmetric partner potentials constructed from (2.9-10.b) and (2.15.a-b) usually have different structures. However, the above potentials are called shape-invariant if $V_2(x)$ has the same functional dependence on the coordinate as $V_1(x)$ and differ only in some parameters [21]. The shape-invariant potentials are defined by the relationship

$$\begin{aligned}
V_2(x, a_0) - V_1(x, a_1) &= 2U(x)\,W'(x, a_0) - U(x)\,U''(x) \\
&= \mathcal{R}(a_0)\,, && (2.16)
\end{aligned}$$

where $a_1 = y(a_0)$ is a function of parameters a_0 and $\mathcal{R}(a_0)$ is independent of variable x. As a consequence, it can be shown that the discrete energy spectrum of $V_1(x)$ can be written as [21]

$$E_n = \sum_{k=0}^{n-1} \mathcal{R}(a_k)\,, \tag{2.17}$$

where the parameter a_k is generated by the consecutive application of the function $y(x)$, i.e.

$$a_k \equiv y^{(k)}(a_0) = \underbrace{y \circ y \circ \cdots \circ y(a_0)}_{\text{k-times}}. \tag{2.18}$$

Having determined the bound-state energy spectrum, the ground-state wavefunction of the corresponding Hamiltonian is obtainable by solving the first-order differential equation

$$A_{eff}\psi_0(x) = 0, \tag{2.19}$$

leading to the wavefunction

$$\psi_0(x) = \frac{\mathcal{N}_0}{\sqrt{U(x)}} \exp\left[-\int^z dz \frac{W(z)}{U(z)}\right], \tag{2.20}$$

where $\mathcal{N}_0$ is some normalization coefficient.

3 Exactly solvable potentials with three parameters

The fairly general factorizable form of superpotential $W(x)$ is given by Ref [21]

$$W(x, a_0) = \sum_{i=1}^{s} (k_i + c_i) g_i(x) + \frac{h_i(x)}{k_i + c_i} + f_i(x), \tag{3.1}$$

where $a_0 = (k_1, k_2, \ldots)$, $a_1 = (k_1 + \alpha, k_2 + \beta, \ldots)$ and $c_i, \alpha, \beta, \ldots$ being constants. It may be noted that one can obtain the condition to be fulfilled by the functions $g_i(x), h_i(x)$ and $f_i(x)$ once that (3.1) is used in (2.16). However, Samani and Loran have carried out a simple form to W considering it in the ansatz with one-parameter [23]

$$W(x, a) = ag(x) + \frac{h(x)}{a} + f(x), \tag{3.2}$$

where the functions $g(x), h(x)$ and $f(x)$ are independent of the parameter a.

We shall now point out the specific ansatz carried out by the superpotential that goes into the determination of the effective potentials and its accompanying energy eigenvalues and ground-state wavefunctions. To this end, the "new" superpotential is characterized by one function and three parameters instead of that given by (3.2). To be more precise, we substitute all functions appearing in (3.2) by new parameters and the single parameter by a function, i.e. $g(x) \to \lambda,\ f(x) \to \sigma,\ h(x) \to \rho$ and $a \to \phi(x)$, without making a point of seeking the transformation which connects them.

Therefore the superpotential (3.2) becomes, taking into account (2.14)

$$W(x, \boldsymbol{\lambda}) = \lambda \phi(x) + \frac{\rho}{\phi(x)} + \sigma + \frac{U'(x)}{2}, \tag{3.3}$$

and the effective shape-invariance condition reads as

$$V_{2,eff}(x, \boldsymbol{\lambda}_0) = V_{1,eff}(x, \boldsymbol{\lambda}_1) + \mathcal{R}(a_0), \tag{3.4}$$

with $\boldsymbol{\lambda}_i = (\lambda_i, \sigma_i, \rho_i)$, $i = 0, 1, 2, \ldots$ Henceforth, we will suppose that $\boldsymbol{\lambda}$ coincides with $\boldsymbol{\lambda}_0$.

Inserting (3.3) into (3.4) we obtain

$$\left(\lambda_1^2 - \lambda_0^2\right)\phi^2(x) + 2(\lambda_1\sigma_1 - \lambda_0\sigma_0)\phi(x) + \frac{\rho_1^2 - \rho_0^2}{\phi^2(x)} + \frac{2(\rho_1\sigma_1 - \rho_0\sigma_0)}{\phi(x)} + U(x)(\rho_1 + \pi\rho_0)\frac{\phi'(x)}{\phi^2(x)} - U(x)(\lambda_1 + \lambda_0)\phi'(x) + \sigma_1^2 - \sigma_0^2 + 2(\rho_1\lambda_1 - \rho_0\lambda_0) + \mathcal{R}(\boldsymbol{\lambda}_0) = 0. \tag{3.5}$$

with $\pi = \pm 1$. The positive case ($\pi = 1$) is what one ends up obtaining in (3.5) leading to the well-known results listed and tabulated in Refs. [18,19,21], while the negative case ($\pi = -1$) was added thinking that it leads to more interesting one. In the subsequent developments, we will be interested in the last case.

Then, one way for (3.5) to be consistent and solvable is to separate the constant terms from the functions ; i.e.

$$\left(\lambda_1^2 - \lambda_0^2\right)\phi^2(x) + 2(\lambda_1\sigma_1 - \lambda_0\sigma_0)\phi(x) + \frac{\rho_1^2 - \rho_0^2}{\phi^2(x)} + \frac{2(\rho_1\sigma_1 - \rho_0\sigma_0)}{\phi(x)} + q = U(x)(\lambda_1 + \lambda_0)\phi'(x) - U(x)(\rho_1 - \rho_0)\frac{\phi'(x)}{\phi^2(x)}, \tag{3.6}$$

and

$$\sigma_1^2-\sigma_0^2+2\left(\rho_1\lambda_1-\rho_0\lambda_0\right)+\mathcal{R}\left(\boldsymbol{\lambda}_0\right)=q, \tag{3.7}$$

with $q\neq 0$. The resolution of (3.6) in terms of $\phi(x)$ amounts to comparing these two members. Consequently, it is of primary importance to write a term $U(x)\phi'(x)$ in such a way that the member of left-hand side is identified to that of right-hand side. To this end, three particular cases (solutions) arise, and henceforth, we will name them constant, linear and quadratic solutions and are identified with differential equations, respectively

$$U(x)\phi'(x) = a, \tag{3.8.a}$$

$$U(x)\phi'(x) = a\phi(x)+b, \quad (a<0), \tag{3.8.b}$$

$$U(x)\phi'(x) = a\phi^2(x)+b\phi(x)+c. \tag{3.8.c}$$

where a, b and c are constants. It is obvious that the solutions in $\phi(x)$ can be explicitly carried out by Euler's type integration [24].

3.1 Constant solution : Three-dimensional harmonic oscillator

3.1.1 Superpotential and Effective potential

After a simple integration, the function $\phi(x)$ is given by

$$\phi(x)=a\mu(x)+b, \tag{3.9}$$

where $(a,b)\in\mathbb{R}^2$ and $\mu(x)$ is a function defined as a dimensionless integral

$$\mu(x)=\int^{z}\frac{dz}{U(z)}, \tag{3.10}$$

and which will appear frequently in subsequent subsections.

Using both (3.3) and (2.15.a) we obtain, respectively, the superpotential and the corresponding effective potential

$$W(x,\boldsymbol{\lambda})=\lambda\left(a\mu(x)+b\right)+\frac{\rho}{a\mu(x)+b}+\sigma+\frac{U'(x)}{2}, \tag{3.11}$$

$$\begin{aligned} V_{eff}(x,\boldsymbol{\lambda}) &= \lambda^2\left(a\mu(x)+b\right)^2+\frac{\rho(\rho+a)}{\left(a\mu(x)+b\right)^2}+2\lambda\sigma\left(a\mu(x)+b\right) \\ &\quad +\frac{2\sigma\rho}{a\mu(x)+b}+\sigma^2+2\lambda\rho-a\lambda. \end{aligned} \tag{3.12}$$

It is obvious that if the $\sigma = b = 0$ constraint holds, then the effective potential (3.12) is related to shape-invariant three-dimensional harmonic oscillator potential

$$V_{eff}^{(\mathrm{H.O})}(x, \boldsymbol{\lambda}) = \lambda^2 a^2 \mu^2(x) + \frac{\rho(\rho + a)}{a^2 \mu^2(x)} + 2\lambda\rho - a\lambda. \tag{3.13}$$

3.1.2 Energy eigenvalues

The energy eigenvalues can be calculated algebraically from (3.7). Indeed, inserting (3.8.a) into (3.6) we obtain

$$\begin{aligned}
\lambda_1^2 - \lambda_0^2 &= \lambda_1\sigma_1 - \lambda_0\sigma_0 = \rho_1\sigma_1 - \rho_0\sigma_0 = 0, && \text{(3.14.a)}\\
\rho_1^2 - \rho_0^2 &= -a(\rho_1 - \rho_0), && \text{(3.14.b)}\\
q &= a(\lambda_1 + \lambda_0) \neq 0. && \text{(3.14.c)}
\end{aligned}$$

Solving (3.14) gets $\rho_1 = -(\rho_0 + a)$, $\lambda_1 = \lambda_0$ and $\sigma_1 = \sigma_0$, which satisfies the recursion relations

$$\rho_k = (-1)^k \rho_0 - \frac{a}{2}\left(1 - (-1)^k\right), \quad \lambda_k = \lambda_0, \quad \sigma_k = \sigma_0, \tag{3.15}$$

where $k = 0, 1, 2, \ldots$ Thus the energy eigenvalues are given, taking into consideration (2.17)

$$\begin{aligned}
\mathcal{E}_n &= \sum_{k=0}^{n-1} \mathcal{R}(\lambda_k, \sigma_k, \rho_k)\\
&= \sum_{k=0}^{n-1} 4\lambda_k(\rho_k + a)\\
&= 4a\sum_{k=0}^{n-1}\lambda_0 + 4\sum_{k=0}^{n-1}\lambda_0\left[(-1)^k\rho_0 - \frac{a}{2}\left(1 - (-1)^k\right)\right]\\
&= 2a\lambda_0 n + \lambda_0(a + 2\rho_0)(1 - (-1)^n). && \text{(3.16)}
\end{aligned}$$

This spectrum presents some features with that obtained in formula (10) in Ref. [25]. Hence the negative case is allowed to generate new shape-invariant potentials with richer new energy spectra of the bound-states with a non-equidistant spectrum.

3.1.3 Ground-state wavefunction

Since a particle is constrained to move in three-dimensional Harmonic Oscillator, we set $\sigma_k = b = 0 \ \ (k = 0, 1, 2, \ldots)$, $\lambda_0 = \dfrac{\omega}{2a}$ and $\rho_0 = al \ \ (l < 0)$. Then, keeping in mind (3.10), the ground-state wavefunction is calculated straightforwardly using (2.19), we finally get

$$\begin{aligned}
\psi_0(x) &= \frac{\mathcal{N}_0}{\sqrt{U(x)}} \exp\left[-\lambda_0 a \int^z dz \frac{\mu(z)}{U(z)} - \frac{\rho_0}{a} \int^z \frac{dz}{U(z)\,\mu(z)} - \frac{1}{2} \int^z \frac{dU(z)}{U(z)} \right] \\
&= \frac{\mathcal{N}_0}{U(x)} \mu^{-l}(x) \exp\left[-\frac{\omega}{4} \mu^2(x) \right]. \qquad (3.17)
\end{aligned}$$

3.2 Linear solution : Morse potential

3.2.1 Superpotential and Effective potential

In this case, the function $\phi(x)$ is given by

$$\phi(x) = \frac{1}{a} \left(b - \exp\left[-a\mu(x) \right] \right). \qquad (3.18)$$

with $a \neq 0$. The corresponding superpotential and the effective potential read as

$$W(x, \boldsymbol{\lambda}) = \frac{\lambda}{a} \left(b - \exp\left[-a\mu(x) \right] \right) + \frac{a\rho}{b - \exp\left[-a\mu(x) \right]} + \sigma + \frac{U'(x)}{2}, \quad (3.19)$$

$$\begin{aligned}
V_{eff}(x, \boldsymbol{\lambda}) &= \frac{\lambda^2}{a^2} \left(b - \exp\left[-a\mu(x) \right] \right)^2 + \frac{a^2 \rho (\rho + b)}{\left(b - \exp\left[-a\mu(x) \right] \right)^2} \\
&\quad + \frac{a\rho (2\sigma - a)}{b - \exp\left[-a\mu(x) \right]} + \lambda \left(\frac{2\sigma}{a} + 1 \right) \left(b - \exp\left[-a\mu(x) \right] \right) \\
&\quad + \sigma^2 + 2\lambda\rho - b\lambda. \qquad (3.20)
\end{aligned}$$

From (3.20), if the $\rho = b = 0$ constraint holds then the effective potential is related to shape-invariant Morse potential

$$V_{eff}^{(\mathrm{Morse})}(x, \boldsymbol{\lambda}) = \frac{\lambda^2}{a^2} \, \mathrm{e}^{-2a\mu(x)} - \lambda \left(\frac{2\sigma}{a} + 1 \right) \mathrm{e}^{-a\mu(x)} + \sigma^2. \qquad (3.21)$$

3.2.2 Energy eigenvalues

Inserting (3.8.b) into (3.6), we get the system of parametric equations according to

$$\begin{aligned} \lambda_1^2-\lambda_0^2 &= 0, && (3.22.\mathrm{a})\\ 2\left(\lambda_1\sigma_1-\lambda_0\sigma_0\right) &= a\left(\lambda_1+\lambda_0\right), && (3.22.\mathrm{b})\\ \rho_1^2-\rho_0^2 &= -b\left(\rho_1-\rho_0\right), && (3.22.\mathrm{c})\\ 2\left(\rho_1\sigma_1-\rho_0\sigma_0\right) &= -a\left(\rho_1-\rho_0\right), && (3.22.\mathrm{d})\\ q &= b\left(\lambda_1+\lambda_0\right)\neq 0. && (3.22.\mathrm{e}) \end{aligned}$$

and its solution lead to the following recursion relations

$$\rho_k=(-1)^k\rho_0-\frac{b}{2}\left(1-(-1)^k\right),\quad \lambda_k=\lambda_0,\quad \sigma_k=\sigma_0+ak. \tag{3.23}$$

The energy eigenvalues can be calculated easily

$$\begin{aligned} \mathcal{E}_n &= \sum_{k=0}^{n-1}\mathcal{R}\left(\lambda_k,\sigma_k,\rho_k\right)\\ &= \sum_{k=0}^{n-1}\left(4b\lambda_k-2a\sigma_k+4b\rho_k-a^2\right)\\ &= n\left(2b\lambda_0-2a\sigma_0-a^2n\right)+\lambda_0\left(2\rho_0+b\right)\left(1-(-1)^n\right) \end{aligned} \tag{3.24}$$

Again, we obtain a richer spectra for the Morse potential as was done in (3.16) for the three-dimensional harmonic oscillator. The energy eigenvalues corresponding to the effective potential (3.21) can be obtained by setting the $\rho_0=b=0$ constraint

$$\mathcal{E}_n^{(\mathrm{Morse})}=-a^2\left(\frac{\sigma_0}{a}+n\right)^2+\sigma_0^2. \tag{3.25}$$

3.2.3 Ground-state wavefunction

The ground-state wavefunction $\psi_0(x)$ associated to the energy eigenvalues (3.21) is given after integration

$$\begin{aligned} \psi_0(x) &= \frac{\mathcal{N}_0}{\sqrt{U(x)}}\exp\left[\frac{\lambda_0}{a}\int^z dz\frac{\mathrm{e}^{-a\mu(z)}}{U(z)}-\frac{a}{2}\int^z\frac{dz}{U(z)}-\frac{1}{2}\int^z\frac{dU(z)}{U(z)}\right]\\ &= \frac{\mathcal{N}_0}{U(x)}\,\mathrm{e}^{-a\mu(x)/2}\exp\left[-\frac{\lambda_0}{a^2}\,\mathrm{e}^{-a\mu(z)}\right]. \end{aligned} \tag{3.26}$$

3.3 Quadratic solution : Pöschl-Teller (I, II) and three dimensional Coulomb potentials

Integrating (3.8.c) in terms of the function $\phi(x)$, it is straightforward to obtain (see 2.172 of Ref. [24])

$$\int^{\phi(x)} \frac{d\xi}{a\xi^2+b\xi+c} = \frac{2}{\sqrt{\Delta}}\arctan\frac{b+2a\phi(x)}{\sqrt{\Delta}} \quad , \quad \Delta>0 \quad (3.27.a)$$

$$= \frac{-2}{\sqrt{-\Delta}}\operatorname{arctanh}\frac{b+2a\phi(x)}{\sqrt{-\Delta}}, \quad \Delta<0 \quad (3.27.b)$$

$$= \frac{-2}{b+2a\phi(x)} \quad , \quad \Delta=0 \quad (3.27.c)$$

where $\Delta=-b^2+4ac$. It is obvious that the integral appearing up is equal to a dimensionless mass integral, i.e. $\mu(x)$.

We will see in subsequent developments that the first two cases are related to trigonometric and hyperbolic Pöschl-Teller potentials, respectively, while the last case is associated to the three-dimensional Coulomb potential.

3.3.1 Pöschl-Teller I and II

Superpotential and Effective potential The solution in terms of $\phi(x)$ is given through (3.27.a) by $\phi(x)=\frac{\sqrt{\Delta}}{2a}\tan\frac{\sqrt{\Delta}}{2}\mu(x)-\frac{b}{2a}$, the superpotential and the effective potential read as

$$W(x,\boldsymbol{\lambda})=\lambda\left(\frac{\sqrt{\Delta}}{2a}\tan\frac{\sqrt{\Delta}}{2}\mu(x)-\frac{b}{2a}\right)+\frac{\rho}{\frac{\sqrt{\Delta}}{2a}\tan\frac{\sqrt{\Delta}}{2}\mu(x)-\frac{b}{2a}}+\sigma+\frac{U'(x)}{2}. \quad (3.28)$$

$$\begin{aligned} V_{eff}(x,\boldsymbol{\lambda}) = {} & \lambda^2\left(\frac{\sqrt{\Delta}}{2a}\tan\frac{\sqrt{\Delta}}{2}\mu(x)-\frac{b}{2a}\right)^2+\frac{\rho^2}{\left(\frac{\sqrt{\Delta}}{2a}\tan\frac{\sqrt{\Delta}}{2}\mu(x)-\frac{b}{2a}\right)^2} \\ & +2\lambda\sigma\left(\frac{\sqrt{\Delta}}{2a}\tan\frac{\sqrt{\Delta}}{2}\mu(x)-\frac{b}{2a}\right)+\frac{2\rho\sigma}{\frac{\sqrt{\Delta}}{2a}\tan\frac{\sqrt{\Delta}}{2}\mu(x)-\frac{b}{2a}} \\ & -\frac{\frac{\lambda\Delta}{4a}}{\cos^2\frac{\sqrt{\Delta}}{2}\mu(x)}+\frac{\frac{\rho\Delta}{4a}}{\cos^2\frac{\sqrt{\Delta}}{2}\mu(x)\left(\frac{\sqrt{\Delta}}{2a}\tan\frac{\sqrt{\Delta}}{2}\mu(x)-\frac{b}{2a}\right)^2} \\ & +\sigma^2+2\lambda\rho. \end{aligned} \quad (3.29)$$

The $\sigma = b = 0$ constraint leads to the shape-invariant trigonometric Pöschl-Teller potential. Indeed, taking into account $\Delta = 4ac > 0$, the effective potential is reduced to

$$V_{eff}^{(\mathrm{Trig.})}(x) = \frac{\lambda c\left(\frac{\lambda}{a} - 1\right)}{\cos^2 \sqrt{ac}\mu(x)} + \frac{\rho a\left(\frac{\rho}{c} + 1\right)}{\sin^2 \sqrt{ac}\mu(x)} - \left(\lambda\sqrt{\frac{c}{a}} - \rho\sqrt{\frac{a}{c}}\right)^2. \quad (3.30)$$

The hyperbolic Pöschl-Teller superpotential and effective potential are obtainable once the substitution $\sqrt{\Delta} \to i\sqrt{-\Delta}$ is made. As a consequence of this, the superpotential and the effective potential become respectively

$$\begin{aligned} W(x, \boldsymbol{\lambda}) &= -\lambda\left(\frac{\sqrt{-\Delta}}{2a}\tanh\frac{\sqrt{-\Delta}}{2}\mu(x) + \frac{b}{2a}\right) - \frac{\rho}{\frac{\sqrt{-\Delta}}{2a}\tanh\frac{\sqrt{-\Delta}}{2}\mu(x) + \frac{b}{2a}} \\ &\quad +\sigma + \frac{U'(x)}{2}. \end{aligned} \quad (3.31)$$

$$\begin{aligned} V_{eff}(x, \boldsymbol{\lambda}) &= \lambda^2\left(\frac{\sqrt{-\Delta}}{2a}\tanh\frac{\sqrt{-\Delta}}{2}\mu(x) + \frac{b}{2a}\right)^2 + \frac{\rho^2}{\left(\frac{\sqrt{-\Delta}}{2a}\tanh\frac{\sqrt{-\Delta}}{2}\mu(x) + \frac{b}{2a}\right)^2} \\ &\quad -2\lambda\sigma\left(\frac{\sqrt{-\Delta}}{2a}\tanh\frac{\sqrt{-\Delta}}{2}\mu(x) + \frac{b}{2a}\right) - \frac{2\rho\sigma}{\frac{\sqrt{-\Delta}}{2a}\tanh\frac{\sqrt{-\Delta}}{2}\mu(x) + \frac{b}{2a}} \\ &\quad -\frac{\frac{\lambda\Delta}{4a}}{\cosh^2\frac{\sqrt{-\Delta}}{2}\mu(x)} + \frac{\frac{\rho\Delta}{4a}}{\cosh^2\frac{\sqrt{-\Delta}}{2}\mu(x)\left(\frac{\sqrt{-\Delta}}{2a}\tanh\frac{\sqrt{-\Delta}}{2}\mu(x) + \frac{b}{2a}\right)^2} \\ &\quad +\sigma^2 + 2\lambda\rho. \end{aligned} \quad (3.32)$$

As in the trigonometric case, the $\sigma = b = 0$ constraint leads to the shape-invariant hyperbolic Pöschl-Teller potential. Taking into account $\Delta = 4ac < 0$, the hyperbolic effective potential is reduced to

$$V_{eff}^{(\mathrm{Hyp.})}(x, \boldsymbol{\lambda}) = \frac{\lambda c\left(\frac{\lambda}{a} - 1\right)}{\cosh^2 \sqrt{-ac}\mu(x)} - \frac{\rho a\left(\frac{\rho}{c} + 1\right)}{\sinh^2 \sqrt{-ac}\mu(x)} - \left(\lambda\sqrt{\frac{c}{a}} - \rho\sqrt{\frac{a}{c}}\right)^2. \quad (3.33)$$

Energy eigenvalues Substituting (3.8.c) in (3.5) we obtain the system of parametric equations

$$\lambda_1^2 - \lambda_0^2 = a(\lambda_1 + \lambda_0), \qquad (3.34.a)$$

$$2(\lambda_1\sigma_1 - \lambda_0\sigma_0) = b(\lambda_1 + \lambda_0), \qquad (3.34.b)$$

$$\rho_1^2 - \rho_0^2 = -c(\rho_1 - \rho_0), \qquad (3.34.c)$$

$$2(\rho_1\sigma_1 - \rho_0\sigma_0) = -b(\rho_1 - \rho_0), \qquad (3.34.d)$$

$$q = c(\lambda_1 + \lambda_0) - a(\rho_1 - \rho_0) \neq 0. \qquad (3.34.e)$$

From (3.34), we deduce the following recursion relations

$$\rho_k = (-1)^k \rho_0 - \frac{c}{2}\left(1 - (-1)^k\right), \quad \lambda_k = \lambda_0 + ak, \quad \sigma_k = \sigma_0 + bk. \qquad (3.35)$$

then the energy eigenvalues are given by

$$\begin{aligned}
\mathcal{E}_n &= \sum_{k=0}^{n-1} \mathcal{R}(\lambda_k, \sigma_k, \rho_k) \\
&= \sum_{k=0}^{n-1} (4c\lambda_k + 4a\rho_k + 4\lambda_k\rho_k - 2b\sigma_k + \Delta) \\
&= \left(a\rho_0 + 2\lambda_0\rho_0 + c\lambda_0 + \frac{ac}{2}\right)(1 - (-1)^n) - n(2a\rho_0 + ac)(-1)^n \\
&\quad + n(2c\lambda_0 + ac - 2b\sigma_0) + n^2\left(ac - b^2\right) \qquad (3.36)
\end{aligned}$$

Taking into account the constraint $\sigma_k = b = 0$ as well as the sign of the discriminant Δ[1], the energy spectra of the bound-states for trigonometric and hyperbolic Pöschl-Teller potentials are

$$\mathcal{E}_n^{(\text{Trig.})} = \left[\frac{1}{2} + \frac{\lambda_0}{a} + n - \left(a\rho_0 + \frac{1}{2}\right)(-1)^n\right]^2 - \left(\frac{\lambda_0}{a} - a\rho_0\right)^2. \qquad (3.37)$$

$$\mathcal{E}_n^{(\text{Hyp.})} = -\left[\frac{1}{2} + \frac{\lambda_0}{a} + n + \left(a\rho_0 - \frac{1}{2}\right)(-1)^n\right]^2 + \left(\frac{\lambda_0}{a} + a\rho_0\right)^2. \qquad (3.38)$$

The energy eigenvalues thus obtained present, as for the case of three-dimensional harmonic oscillator, some similar features with that obtained in formulas (22) and (29) in Ref. [25], respectively.

[1] This means that $\operatorname{sign} a = \operatorname{sign} c$ for $\Delta > 0$ and $\operatorname{sign} a = -\operatorname{sign} c$ for $\Delta < 0$.

Ground-state wavefunction The ground-state wavefunction $\psi_0(x)$ associated to the trigonometric and hyperbolic Pöschl-Teller potentials (3.30) and (3.33) are given respectively

$$\psi_0^{(\mathrm{Trig.})}(x) = \frac{\mathcal{N}_0}{U(x)} \cos^{\lambda/a} \sqrt{ac}\mu(x) \sin^{-\rho/c} \sqrt{ac}\mu(x) \,. \tag{3.39}$$

$$\psi_0^{(\mathrm{Hyp.})}(x) = \frac{\mathcal{N}_0}{U(x)} \cosh^{\lambda/a} \sqrt{-ac}\mu(x) \sinh^{\rho/c} \sqrt{-ac}\mu(x) \,. \tag{3.40}$$

3.3.2 Three-dimensional Coulomb potential

Superpotential and Effective potential Using (3.27.c), the function $\phi(x)$ becomes

$$\phi(x) = -\left[\frac{2 + b\mu(x)}{2a\mu(x)}\right], \tag{3.41}$$

and the corresponding superpotential and the effective potential read

$$W(x, \boldsymbol{\lambda}) = -\lambda \left(\frac{2 + b\mu(x)}{2a\mu(x)}\right) - \frac{2a\rho\mu(x)}{2 + b\mu(x)} + \sigma + \frac{U'(x)}{2}, \tag{3.42}$$

$$\begin{aligned} V_{eff}(x, \boldsymbol{\lambda}) &= \lambda^2 \left(\frac{2 + b\mu(x)}{2a\mu(x)}\right)^2 + \frac{4a^2\rho^2\mu^2(x)}{(2 + b\mu(x))^2} - \lambda\rho \left(\frac{2 + b\mu(x)}{2a\mu(x)}\right) \\ &\quad - \frac{4a\rho\sigma\mu(x)}{2 + b\mu(x)} - \frac{\lambda}{a\mu^2(x)} + \frac{4a\rho}{(2 + b\mu(x))^2} + 2\rho\lambda + \sigma^2. \end{aligned} \tag{3.43}$$

At first sight, the $\rho = 0$ constraint leads to the shape-invariant three-dimensional Coulomb potential

$$V_{eff}^{(\mathrm{Cb.})}(x, \boldsymbol{\lambda}) = \frac{\lambda}{a}\left(\frac{\lambda}{a} - 1\right)\frac{1}{\mu^2(x)} + \frac{2\lambda}{a}\left(\frac{b\lambda}{2a} - \sigma\right)\frac{1}{\mu(x)} + \left(\frac{b\lambda}{2a} - \sigma\right)^2. \tag{3.44}$$

Energy eigenvalues Since we are dealing with three-dimensional Coulomb potential, we will assume that $\frac{\lambda}{a}\left(\frac{\lambda}{a} - 1\right) = l(l+1)$ and $\frac{2\lambda}{a}\left(\frac{b\lambda}{2a} - \sigma\right) = -Ze^2$, it is straightforward to get

$$\lambda = a(l+1) \;, \qquad \sigma = \frac{b}{2}(l+1) + \frac{Ze^2}{2(l+1)}, \tag{3.45}$$

where l is the angular momentum quantum number, Z the atomic number and e the electronic charge. Here the parameters λ and σ coincide with λ_0 and σ_0, respectively. As a consequence to (3.45), the recursion relations (3.35) give

$$\lambda_k = a\left(k+l+1\right) \ , \qquad \sigma_k = \frac{b}{2}\left(l+1\right) + \frac{Ze^2}{2\left(l+1\right)} + bk. \tag{3.46}$$

for $k = 0, 1, 2, \ldots$ Since the potential (3.44) is obtained from the restriction $\Delta = \rho_k = 0$, this requires that the coefficients a, b and c are nonzero and $\rho_k = \left(-1\right)^k \rho_0 - \frac{c}{2}\left(1 - \left(-1\right)^k\right) = 0$ imposes the condition that k should be even, i.e. $k = 2p$ with $p \in \mathbb{N}$. As defined above in (3.36) and taking into account the last restriction, the shape-invariance condition function $\mathcal{R}\left(\lambda_k, \sigma_k, \rho_k\right)$ satisfies

$$\begin{aligned} \mathcal{R}\left(\lambda_{kl}, \sigma_{kl}\right) &= 4c\lambda_{kl} - 2b\sigma_{kl} \\ &= 4ac\left(k+l+1\right) - 2b\left[\frac{b\left(l+1\right)}{2} + \frac{Ze^2}{2\left(l+1\right)} + bk\right] \\ &= -\left(b^2 k + \frac{Ze^2 b}{l+1}\right). \end{aligned} \tag{3.47}$$

The energy eigenvalues can now be obtained from equations (2.17) and (3.47). By inserting $k = 2p$ in (3.47), the eigenvalues can be rewritten as

$$\begin{aligned} \mathcal{E}_{N,l}^{(\mathrm{Cb.})} &= \sum_{k=0}^{N-1} \mathcal{R}\left(\lambda_{kl}, \sigma_{kl}\right) \\ &= -\sum_{p=0}^{\mathrm{int}\left[\frac{N-1}{2}\right]} \left(2b^2 p + \frac{Ze^2 b}{l+1}\right) \\ &= \frac{-b}{l+1}\left(1 + \mathrm{int}\left[\frac{N-1}{2}\right]\right)\left(Ze^2 + b\left(l+1\right)\mathrm{int}\left[\frac{N-1}{2}\right]\right) \end{aligned} \tag{3.48}$$

where $\mathrm{int}\left[\frac{N-1}{2}\right]$ is the greatest integer not larger than $\frac{N-1}{2}$. By definition, $\mathrm{int}\left[\frac{x}{2}\right] = \frac{x}{2}$ for x even and $\frac{x-1}{2}$ for x odd. Here, both l and N are

related by the relationship

$$\begin{aligned}
\mathrm{int}\left[\frac{N-1}{2}\right] &= \frac{N-1-s}{2} \\
&= \frac{2n-1-(2l+1)}{2} \\
&= n-l-1 \\
&= n_r. \qquad (3.49)
\end{aligned}$$

with $s = 2l+1$, n is the principal quantum number and n_r a quantum number which denotes the number of radial nodes for the wavefunction. Here, we have assumed $N = 2n$ such that the number of bound-state levels is equal to $\mathrm{int}\left[n + \frac{1}{2}\right] = n$ for $n \neq 0$. However, we can note, from (3.49), that the angular momentum quantum number l fulfills the condition $l \neq -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$

If we make the replacement $b = \frac{Ze^2}{2(n_r+l+1)} + \frac{Ze^2}{2(l+1)}$, then the energy eigenvalues (3.48) become

$$\mathcal{E}_{n_r,l}^{(\mathrm{Cb.})} = \frac{-Z^2e^4(1+n_r)}{4(n_r+l+1)^2(l+1)^2}(n_r+2l+2)\left(n_r^2+2n_r+2(l+1)(n_r+1)\right). \qquad (3.50)$$

For large values of l and n_r, we can make the approximation $l \sim l+1$ and $n_r \sim n_r+1$ leading to impose that both quantum numbers take all integral values from 0 to $l_{\max}$ and $n_{r\,\max}$, respectively, i.e. $l = 0, 1, 2, \ldots, l_{\max}$ and $n_r = 0, 1, 2, \ldots, n_{r\,\max}$. Thus, the energy eigenvalues can now be obtained from (3.50) as

$$\begin{aligned}
\mathcal{E}_{n_r,l}^{(\mathrm{Cb.})} &= -\frac{Z^2e^4}{4(n_r+l+1)^2(l+1)^2}n_r^2(n_r+2l+2)^2 \\
&= -\left[\frac{Ze^2-\kappa F(n_r,l)}{2\ (n_r+l+1)}\right]^2, \qquad (3.51)
\end{aligned}$$

with $F(n_r,l) = n_r^2 + (l+1)(2n_r+1)$ and $\kappa = \frac{Ze^2}{l+1}$. From general considerations, it is evident that the spectrum of negative eigenvalues of the energy will be discrete, while that of the positive eigenvalues will be continuous. It

follows from (3.51) that the function $F(n_r, l)$ is fulfilled by the condition

$$F(n_r, l) = n_r^2 + (l+1)(2n_r + 1) < \frac{Ze^2}{\kappa}. \tag{3.52}$$

The energy spectra given in (3.51) have already been established by Quesne and Tkachuk [18,19].

Ground-state wavefunction Let us now complete this algebraic determination of energy eigenvalues by a construction of the corresponding "radial" ground-state wavefunction $\psi_{0l}(x)$. Using the last considerations, the superpotential can be rewritten as $W_l(x) = -\frac{l+1}{\mu(x)} + \frac{Ze^2}{2(l+1)} + \frac{U'(x)}{2}$ and the "radial" ground-state wavefunction is given through (2.20) by

$$\psi_{0l}(x) = \frac{\mathcal{N}_0}{U(x)} \mu^{l+1}(x) \exp\left[-\frac{Ze^2}{2\ (l+1)}\mu(x)\right]. \tag{3.53}$$

4 Conclusion

In the present paper, we have generated four "new" solvable, real and shape-invariant potentials simply by judicious applications of the supersymmetric quantum mechanics formalism and shape-invariant potentials. In all cases, we have derived the effective potentials as well as their accompanying energy spectra of bound-states and ground-state wavefunctions. However, the new and the important contribution of this paper is to point out how the simple fact of going from the positive case ($\pi = 1$), characterized by an equidistant spectra, to the negative one ($\pi = -1$), allows to generate a new shape-invariant three-dimensional harmonic oscillator, Morse and Pöschl-Teller (I and II) potentials with non-equidistant spectra (3.16), (3.24) and (3.36), respectively, while the three-dimensional Coulomb potential has only a finite number of bound-states (3.51) in contrast with the standard coulomb problem.

To conclude, the supersymmetric quantum mechanics can also be a useful machinery for the treatment of wide classes of potentials. It may be possible to explore, in the context of position-dependent effective mass within the framework of non compact **SO** $(2,2)$ Lie algebra, all shape-invariant potentials listed and tabulated in Refs. [19,21], respectively. The works are in progress and will be deferred to later publication.

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